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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM NO. 1349

ON A CLASS OF EXACT SOLUTIONS OF THE EQUATIONS OF MOTION
OF A VISCOUS FLUID*

By V. I. Yatseyev

The general solution is obtained herein of the equations of motion of a viscous fluid in which the velocity field is inversely proportional to the distance from a certain point. Some particular cases of such motion are investigated.

1. The motion of a viscous fluid with velocity field and pressure in spherical coordinates can be given by the following expressions:

$$v_r = \frac{F(\theta)}{r} \quad v_\theta = \frac{f(\theta)}{r} \quad v_\phi = 0 \quad \frac{p}{\rho} = \frac{g(\theta)}{r^2} \quad (1)$$

A particular solution of the equations of Navier-Stokes for this case was obtained by Landau (reference 1). In the present paper a general solution is given of the equations of Navier-Stokes for the motion of the class under consideration.

Substituting expressions (1) in the equations of Navier-Stokes and in the equation of continuity yields the following system:

$$F^2 + f^2 - fF' + 2g + \nu \left[F'' + F' \cot \theta - 2f' - 2F - 2f \cot \theta \right] = 0 \quad (2)$$

$$ff' + g' - \nu \left[f'' + f' \cot \theta + 2F' - f(1 + \cot^2 \theta) \right] = 0 \quad (3)$$

$$F + f' + f \cot \theta = 0 \quad (4)$$

Determining F from equation (4) and substituting in equations (2) and (3) give

*"Ob odnom klasse tochnykh reshenii uravnenii dvizheniya vyazkoi zhidkosti." Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki, vol. 20, no. 11, 1950, pp. 1031-1034.

$$f'^2 + ff'' + 3ff' \cot \theta + 2g - \nu \left[f''' + 2f'' \cot \theta - f' (2 + \cot^2 \theta) + f \cot \theta (1 + \cot^2 \theta) \right] = 0 \quad (5)$$

$$ff' + g' + \nu \left[f'' + f' \cot \theta - f (1 + \cot^2 \theta) \right] = 0 \quad (6)$$

Differentiating expression (6)

$$f'^2 + ff'' + g'' + \nu \left[f''' + f'' \cot \theta - 2f' (1 + \cot^2 \theta) + 2f \cot \theta (1 + \cot^2 \theta) \right] = 0 \quad (7)$$

Eliminating the nonlinear terms f'^2 , ff'' , and ff' from equation (5) with the aid of equations (6) and (7) yields a linear equation in the function $g + 2\nu f'$:

$$(g + 2\nu f')'' + 3 \cot \theta (g + 2\nu f')' - 2 (g + 2\nu f') = 0 \quad (8)$$

the general solution of which is in the form

$$g + 2\nu f' = 2\nu^2 \frac{b \cos \theta - a}{\sin^2 \theta} \quad (9)$$

where $2\nu^2 a$ and $2\nu^2 b$ are constants of integration.

Integrating equation (6)

$$f^2 + 2g + 2\nu (f' + f \cot \theta) = - 2\nu^2 c \quad (10)$$

where $2\nu^2 c$ is the constant of integration.

The function $g(\theta)$ is eliminated from equations (9) and (10) to give an equation of the Riccati type for the function f :¹

$$f' = \frac{1}{2\nu} f^2 + f \cot \theta + 2\nu \left(\frac{b \cos \theta - a}{\sin^2 \theta} + \frac{c}{2} \right) \quad (11)$$

¹After sending the manuscript to press the author obtained from L. D. Landau a communication on the work of N. Slezkin (reference 2) in which he arrived at the same equation by a different method.

The substitution

$$f = -2\nu\chi'(\theta)/\chi(\theta) \quad (12)$$

reduces equation (11) to the linear equation:

$$\chi'' - \chi' \cot \theta + \left(\frac{b \cos \theta - a}{\sin^2 \theta} + \frac{c}{2} \right) \chi = 0 \quad (13)$$

which by the substitution

$$z = \cos^2 (\theta/2) \quad (14)$$

is transformed into an equation of the Fuchsian type:

$$\frac{d^2\chi}{dz^2} - \frac{a + b - 2(b + c)z + 2cz^2}{4z^2(z - 1)} \chi = 0 \quad (15)$$

The usual computations (reference 3), which are omitted herein, give the general solution of equation (15) as:

$$\chi(\theta) = \left(\cos \frac{\theta}{2} \right)^\gamma \left(\sin \frac{\theta}{2} \right)^{1+\alpha+\beta-\gamma} \left\{ c_1 F \left(\alpha, \beta, \gamma, \cos^2 \frac{\theta}{2} \right) + c_2 F \left(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma, \cos^2 \frac{\theta}{2} \right) \right\} \quad (16)$$

where the parameters of the hypergeometric function α, β, γ (which can also have complex values) are connected with the constants of integration a, b, c by the formulas:

$$\left. \begin{aligned} a &= \gamma^2 - (1 + \alpha + \beta) \gamma + \frac{(\alpha + \beta)^2}{2} - \frac{1}{2} \\ b &= (\alpha + \beta - 1) \gamma - \frac{(\alpha + \beta)}{2} + \frac{1}{2} \\ c &= \frac{(\alpha - \beta)^2 - 1}{2} \end{aligned} \right\} \quad (17)$$

Formulas (4), (9), (16), and (17) give the general solution, depending on the four constants a , b , c , and $A = c_2/c_1$, of the Navier-Stokes equations for the class of motion of a viscous fluid under consideration. The constants of integration a , b , and c are expressed in terms of the corresponding tensor components of the density of the momentum transfer:

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \rho v \left(\frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} \right) \quad (18)$$

Carrying out the computations

$$\left. \begin{aligned} \Pi_{\varphi\varphi} &= \frac{2v^2\rho}{r^2} \left(\frac{b \cos \theta - a}{\sin^2 \theta} \right) \\ \Pi_{\theta\theta} &= \frac{2v^2\rho}{r^2} \left(\frac{a - b \cos \theta}{\sin^2 \theta} - \frac{c}{2} \right) \\ \Pi_{r\theta} &= \frac{2v^2\rho}{r^2} \left(\frac{c \cos \theta - b}{\sin^2 \theta} \right) \end{aligned} \right\} \quad (19)$$

The streamlines are determined by the equation:

$$dr/v_r = r d\theta/v_\theta \quad (20)$$

the integration of which gives

$$\text{const}/r = f \sin \theta \quad (21)$$

2. Attention is now given to two particular examples for which the equation of Fuchs degenerates.

(a) Equation (15) has only one regular singular point, $z = \infty$. In this case

$$a = b = c = 0 \quad (22)$$

and therefore by equations (19)

$$\Pi_{\varphi\varphi} = \Pi_{\theta\theta} = \Pi_{r\theta} = 0 \quad (23)$$

The particular solution of equation (15)

$$\chi(\theta) = 2z - 1 - A \quad (24)$$

leads by formulas (4), (9), and (12) to the solution found by Landau:

$$F(\theta) = 2v \left[\frac{A^2 - 1}{(A - \cos \theta)^2} - 1 \right] \quad f(\theta) = \frac{2v \sin \theta}{\cos \theta - A}$$

$$g(\theta) = 4v^2 \frac{1 - A \cos \theta}{(\cos \theta - A)^2} \quad (25)$$

This solution is analogous to the problem of a stream flowing out of the end of a thin pipe into a region filled with the same fluid. It is the only regular solution for all values of the angle θ .

(b) Equation (15) has only two regular points $z = 0$ and $z = \infty$. In this case it follows from equation (15) that

$$a = b = c \neq 0 \quad (26)$$

and equation (11) becomes Euler's equation

$$2z^2 (d^2\chi/dz^2) - a\chi = 0 \quad (27)$$

the general solutions of which are

$$\left. \begin{aligned} \chi(\theta) &= e^{x/2} \cosh (nx + A) \text{ for } a > -1/2 \\ \chi(\theta) &= e^{x/2} \cos (nx + A) \text{ for } a < -1/2 \\ \chi(\theta) &= e^{x/2} (1 + Ax) \text{ for } a = -1/2 \end{aligned} \right\} \quad (28)$$

Correspondingly, the following equations are obtained for the function $f(\theta)$:

$$f = 2v \frac{\sin \theta}{1 + \cos \theta} \left\{ n \tanh (nx + A) + 1/2 \right\} \text{ for } a > -1/2 \quad (29)$$

$$f = 2v \frac{\sin \theta}{1 + \cos \theta} \left\{ \frac{1}{2} - n \tan (nx + A) \right\} \text{ for } a < -1/2 \quad (30)$$

$$f = 2v \frac{\sin \theta}{1 + \cos \theta} \left\{ \frac{A}{1 + Ax} + 1/2 \right\} \text{ for } a = -1/2 \quad (31)$$

where $x = \ln (1 + \cos \theta)$, $n = \frac{1}{2} \left| \sqrt{1 + 2a} \right|$. (For $a = 0$, $n = 1/2$ in equation (29) the solution of Landau is again obtained.)

For the solution of equation (29) by formula (4)

$$F(\theta) = -2\nu \left\{ n \tanh (nx + A) + 1/2 \right\} + 2\nu \frac{1 - \cos \theta}{1 + \cos \theta} \frac{n^2}{\cosh^2 (nx + A)} \quad (32)$$

while $g(\theta)$ is determined by formula (9).

The equation of the streamlines is in the form

$$\text{const}/r = (1 - \cos \theta) \left\{ n \tanh (nx + A) + 1/2 \right\} \quad (33)$$

where the values of the constants n and A are determined from the conditions

$$\left. \begin{aligned} f\left(\frac{\pi}{2}\right) &= 2\nu (n \tanh A + 1/2) \\ f\left(\frac{\pi}{2}\right) + F\left(\frac{\pi}{2}\right) &= 2\nu \frac{n^2}{\cosh^2 A} \end{aligned} \right\} \quad (34)$$

The obtained solution corresponds to the problem of the stream flowing from the half line $\theta = \pi$ into a region filled with the same fluid.

For solution (30), the parametric equation for the streamlines is in the form

$$\begin{aligned} \text{const}/r &= (2 - e^x) \left[1/2 - n \tan (nx + A) \right] \\ \theta &= \arccos (e^x - 1) \end{aligned} \quad (35)$$

The function (35) for $\theta \rightarrow \pi$ is a strongly oscillating one. It can therefore be concluded that solution (30) has no physical sense.

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